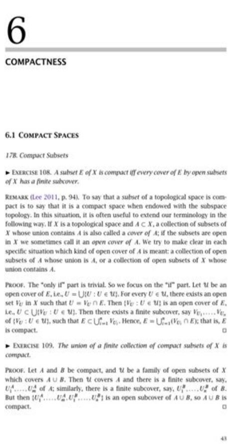

6

COMPACTNESS

6.1 Compact Spaces

17B. Compact Subsets

▸ Exercise 108. *A subset E of X is compact iff every cover of E by open subsets of X has a finite subcover.*

Remark (Lee 2011, p. 94). To say that a *subset* of a topological space is compact is to say that it is a compact space when endowed with the subspace topology. In this situation, it is often useful to extend our terminology in the following way. If X is a topological space and $A \subset X$, a collection of subsets of X whose union contains A is also called a *cover of* A; if the subsets are open in X we sometimes call it an *open cover of* A. We try to make clear in each specific situation which kind of open cover of A is meant: a collection of open subsets of A whose union is A, or a collection of open subsets of X whose union contains A.

Proof. The "only if" part is trivial. So we focus on the "if" part. Let $\mathfrak{U}$ be an open cover of E, i.e., $U = \bigcup\{U : U \in \mathfrak{U}\}$. For every $U \in \mathfrak{U}$, there exists an open set V_U in X such that $U = V_U \cap E$. Then $\{V_U : U \in \mathfrak{U}\}$ is an open cover of E, i.e., $U \subset \bigcup\{V_U : U \in \mathfrak{U}\}$. Then there exists a finite subcover, say $V_{U_1}, \ldots, V_{U_n}$ of $\{V_U : U \in \mathfrak{U}\}$, such that $E \subset \bigcup_{i=1}^{n} V_{U_i}$. Hence, $E = \bigcup_{i=1}^{n}(V_{U_i} \cap E)$; that is, E is compact. □

▸ Exercise 109. *The union of a finite collection of compact subsets of X is compact.*

Proof. Let A and B be compact, and $\mathfrak{U}$ be a family of open subsets of X which covers $A \cup B$. Then $\mathfrak{U}$ covers A and there is a finite subcover, say, $U_1^A, \ldots, U_m^A$ of A; similarly, there is a finite subcover, say, $U_1^B, \ldots, U_n^B$ of B. But then $\{U_1^A, \ldots, U_m^A, U_1^B, \ldots, U_n^B\}$ is an open subcover of $A \cup B$, so $A \cup B$ is compact. □

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